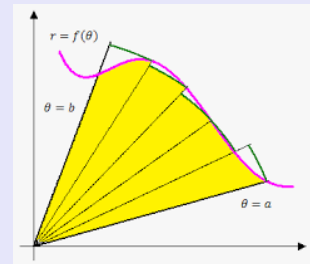
Calculus II

Lecture 1



Feb 19-8:47 AM

Some Review:

1) Evaluate $\lim_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8} = \dfrac{2^2 - 4}{2^3 - 8} = \dfrac{0}{0}$ I.F.

$$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}(x^2 + 2x + 4)}$$

$$= \lim_{x \to 2} \frac{x+2}{x^2 + 2x + 4} = \frac{2+2}{2^2 + 2(2) + 4} = \frac{4}{12}$$

$$= \boxed{\frac{1}{3}}$$

1) $\frac{1}{3}$

2) For $\varepsilon > 0$, find $\delta > 0$ such that $\lim_{x \to 3} (2x-5) = 1$ ✓

$f(x) = 2x-5$

$a = 3$

$L = 1$

verify the limit

$\lim_{x \to 3} (2x-5) = 2(3) - 5 = 1$ ✓

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|2x-5-1| < \varepsilon$ 〃 $|x-3| < \delta$

$|2x-6| < \varepsilon$ 〃 $|x-3| < \delta$

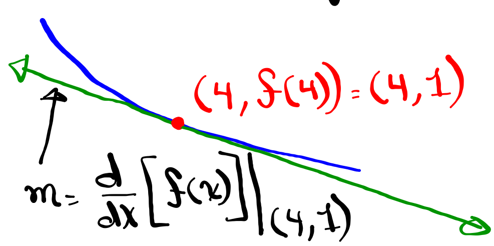
$|2(x-3)| < \varepsilon$ 〃 $|x-3| < \delta$

$2|x-3| < \varepsilon$ 〃 $|x-3| < \delta$

$|x-3| < \frac{\varepsilon}{2}$ 〃 $|x-3| < \delta$

Pick $\delta = \frac{\varepsilon}{2}$

Find the equation of the tangent line to the graph of $f(x) = \frac{2}{\sqrt{x}}$ at $x = 4$.



$y - y_1 = m(x - x_1)$

$f(x) = \frac{2}{\sqrt{x}}$ $\quad f(x) = \frac{2}{x^{1/2}}$ $\quad f(x) = 2x^{-1/2}$

$f'(x) = \cancel{2} \cdot \frac{-1}{\cancel{2}} x^{-3/2} = \frac{-1}{x^{3/2}} = \frac{-1}{x\sqrt{x}}$

$m = f'(4) = \frac{-1}{4\sqrt{4}} = \frac{-1}{8}$

$y - 1 = \frac{-1}{8}(x-4)$

⋮

$\boxed{y = \quad\quad}$

Graph $y = x^3 - 3x$

1) Polynomial $\rightarrow$ Domain $(-\infty, \infty)$

2) Y-Int $(0,0)$

3) x-Int $(0,0)$
$(\pm\sqrt{3}, 0)$

$0 = x^3 - 3x$
$= x(x^2 - 3)$
$x = 0, \quad x = \pm\sqrt{3}$

4) $f(x) = x^3 - 3x$
$f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -[x^3 - 3x] = -f(x)$
what do we conclude when $f(-x) = -f(x)$?
$f(x)$ is odd $\rightarrow$ Symmetry $\rightarrow$ Origin.

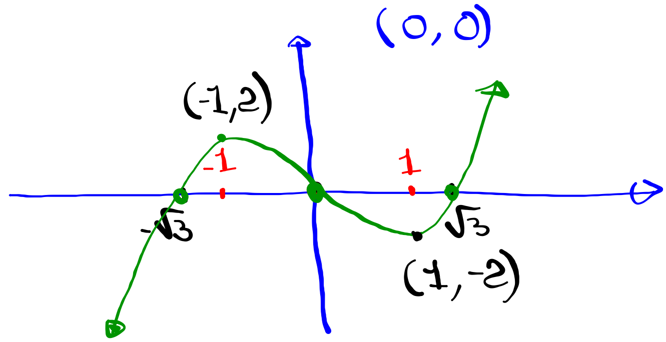
5) $f'(x) = 3x^2 - 3$, $f''(x) = 6x$

6) $f'(x) = 0$
$3x^2 - 3 = 0$
$3(x^2 - 1) = 0$
$x = \pm 1$ C.N.

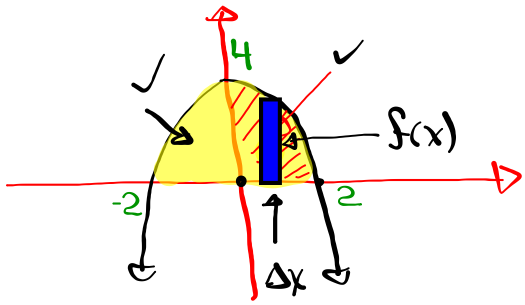
$f''(x) = 0$
$6x = 0$
$x = 0$
P.I.N.

x	$-\infty$		-1		0		1		∞
$f'(x)$		$+$		$-$		$-$		$+$	
$f''(x)$		$-$		$-$		$+$		$+$	
$f(x)$		↗		↘		↘		↗	

Inflection Point
$(0,0)$



Find area below $\underbrace{f(x) = 4 - x^2}$, above x-axis.
Drawing required.

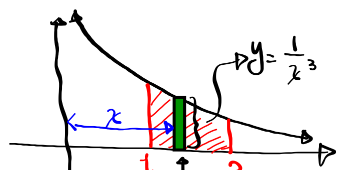


$$A = 2\int_0^2 (4-x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]\Big|_0^2$$

$$= 2\left[\left(4(2) - \frac{2^3}{3}\right) - (0)\right] = \boxed{}$$

Consider the region bounded by $y = \frac{1}{x^3}$, $x=1$, $x=2$, and $y=0$. Find the volume if it is rotated about



1) x-axis

Disk $V = \int_1^2 \pi\left[\frac{1}{x^3}\right]^2 dx$

$$= \pi\int_1^2 x^{-6}\,dx = \pi\left[\frac{x^{-5}}{-5}\right]\Big|_1^2 = \frac{-\pi}{5}\left[\frac{1}{x^5}\right]\Big|_1^2$$

$$= \frac{-\pi}{5}\left[\frac{1}{32} - 1\right] = \boxed{}$$

2) y-axis

Shell

$$V = \int_1^2 2\pi \cdot x \cdot \frac{1}{x^3}\,dx$$

$$= 2\pi\int_1^2 x^{-2}\,dx = 2\pi \cdot \frac{x^{-1}}{-1}\Big|_1^2 = -2\pi\left[\frac{1}{x}\right]\Big|_1^2 = -2\pi\left(\frac{1}{2} - 1\right)$$

$$= -2\pi \cdot \frac{-1}{2}$$

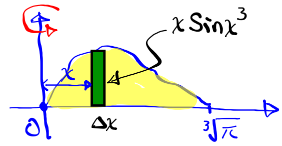
$$= \boxed{\pi}$$

Take the region below $f(x) = x\sin x^3$, above x-axis, from $x=0$ to $x=\sqrt[3]{\pi}$, rotate it about Y-axis, find its volume.

$f(0) = 0\sin 0^3 = 0$

$f(\sqrt[3]{\pi}) = \sqrt[3]{\pi}\ \sin(\sqrt[3]{\pi})^3 = \sqrt[3]{\pi}\ \sin\pi = 0$

$f(x) \geq 0$ on $[0, \sqrt[3]{\pi}]$



Shell

$$V = \int_0^{\sqrt[3]{\pi}} 2\pi \cdot x \cdot x\sin x^3\,dx$$

$$= 2\pi \int_0^{\sqrt[3]{\pi}} x^2 \sin x^3\,dx$$

$u = x^3$ $\quad$ $x=0$ $\quad u=0$

$du = 3x^2\,dx$ $\quad$ $x=\sqrt[3]{\pi}$ $\quad u=\pi$

$\frac{du}{3} = x^2\,dx$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin u\,du$$

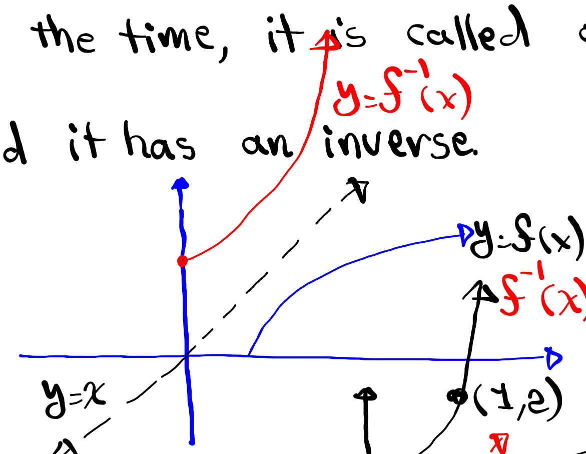
$$= \frac{2\pi}{3}\cdot -\cos u\Big|_0^{\pi} = -\frac{2\pi}{3}\cdot\cos u\Big|_0^{\pi} = -\frac{2\pi}{3}\left[\overset{-1}{\cos\pi} - \overset{1}{\cos 0}\right]$$

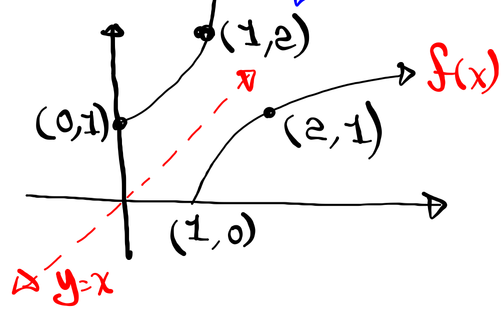
$$= -\frac{2\pi}{3}[-1-1] = \boxed{\frac{4\pi}{3}}$$

If a function is increasing or is decreasing all the time, it is called one-to-one function

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

and it has an inverse.





$f(x) = \sqrt{x-1}$

Domain $[1,\infty)$

How to find $f^{-1}(x)$:

1) Replace $f(x)$ with y.
2) Switch x & y.
3) Solve for y.
4) Replace y with $f^{-1}(x)$.

$f^{-1}(x) = x^2+1$

$f(x) = \sqrt{x-1}$

$y = \sqrt{x-1}$

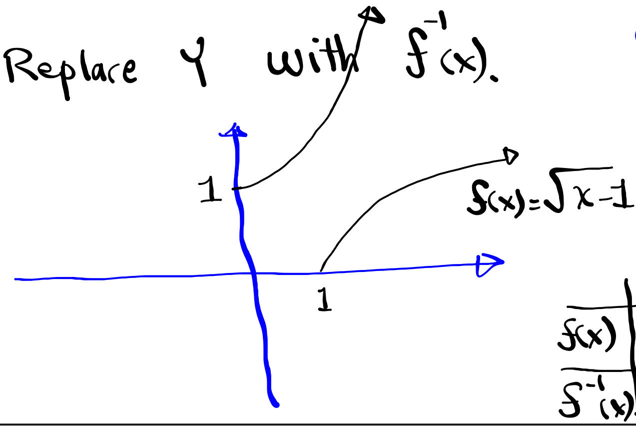
$x = \sqrt{y-1}$

$x^2 = y-1$

$y = x^2+1$

$f^{-1}(x) = x^2+1$

$x \geq 0$



	D	R
$f(x)$	$[1,\infty)$	$[0,\infty)$
$f^{-1}(x)$	$[0,\infty)$	$[1,\infty)$

Find $\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[\sqrt{x-1}\right] = \frac{d}{dx}\left[(x-1)^{1/2}\right]$

$= \frac{1}{2}(x-1)^{-1/2} \cdot 1 = \frac{1}{2\sqrt{x-1}}$

Find $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{d}{dx}\left[x^2+1\right] = 2x$

If $(f\circ g)(x)=x$ and $(g\circ f)(x)=x$, then $f(x)$ and $g(x)$ are inverse of each other.

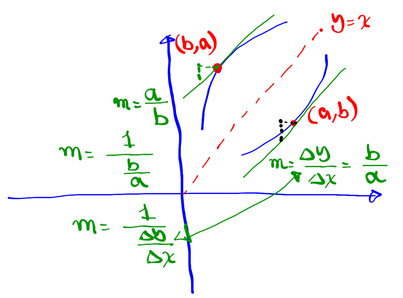
Composition

$f(x)=3x+5$ & $g(x)=\dfrac{x-5}{3}$

$(f\circ g)(x)=f(g(x))=3\,g(x)+5=\cancel{3}\left(\dfrac{x-\cancel{5}}{\cancel{3}}\right)+\cancel{5}=\boxed{x}$

$(g\circ f)(x)=g(f(x))=\dfrac{f(x)-5}{3}=\dfrac{\cancel{3}x+\cancel{5}-\cancel{5}}{\cancel{3}}=\boxed{x}$

$f(x)$ & $g(x)$ are inverse of each other.



Def. of derivative for $f(x)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

ex: $f(x)=x^2$, $a=2$

$f(2)=4$

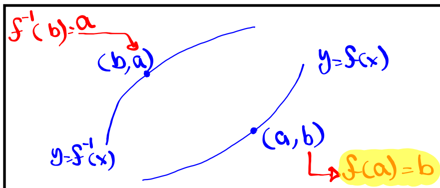
$f'(x)=2x$

$f'(2)=4$ ✓

$$f'(2)=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2}\frac{x^2-4}{x-2}$$

$$=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}$$

$$=\lim_{x\to 2}(x+2)=2+2=\boxed{4}$$



Find $\left(f^{-1}(x)\right)'\Big|_{x=a}$

Recall
$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$(f^{-1})'(a)=\lim_{x\to a}\frac{f^{-1}(x)-f^{-1}(a)}{x-a}$$

$y=f(x)$
$\vdots$
$x=f(y)$

$$=\lim_{y\to b}\frac{y-b}{f(y)-f(b)}$$

$$=\lim_{y\to b}\frac{\frac{y-b}{y-b}}{\frac{f(y)-f(b)}{y-b}}=\lim_{y\to b}\frac{1}{\frac{f(y)-f(b)}{y-b}}$$

$$=\frac{1}{\lim_{y\to b}\frac{f(y)-f(b)}{y-b}}=\frac{1}{f'(b)}$$

We just showed that

$$\boxed{(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}}\qquad=\frac{1}{f'(f(a))}$$

Given $f(x)=x^3$, $a=8$

Find $(f^{-1})'(a)$

$$(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}$$

$$=\frac{1}{f'(2)}=\frac{1}{3(2)^2}$$

$$=\boxed{\frac{1}{12}}\ \checkmark$$

$f^{-1}(x)=\sqrt[3]{x}=x^{1/3}$

$$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{3}x^{-2/3}$$

$$=\frac{1}{3\sqrt[3]{x^2}}$$

$$(f^{-1})'(8)=\frac{1}{3\sqrt[3]{8^2}}=\frac{1}{12}\ \checkmark$$

Given $f(x)=\frac{1}{x-1}=(x-1)^{-1}$, $x>1$, $f'(x)=\frac{-1}{(x-1)^2}$

$(f^{-1})'(2)$

we learned

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(\frac{3}{2})}$$

$$=\frac{1}{\frac{-1}{(\frac{3}{2}-1)^2}}$$

$$=\frac{1}{\frac{-1}{\frac{1}{4}}}=\boxed{\frac{-1}{4}}\checkmark$$

$f(x)=\frac{1}{x-1}$

$y=\frac{1}{x-1}$

$x=\frac{1}{y-1}$

$x(y-1)=1$

$xy-x=1$

$y=\frac{x+1}{x}$

$f^{-1}(x)=\frac{x+1}{x}$

$f^{-1}(x)=1+\frac{1}{x}$

$(f^{-1})'=0+\frac{-1}{x^2}=\frac{-1}{x^2}$

$(f^{-1})'(2)=\boxed{\frac{-1}{4}}\checkmark$

$f(x)=x^3+3\sin x+2\cos x$,

$(f^{-1})'(2)$

$f'(x)=3x^2+3\cos x-2\sin x$

$f^{-1}(2)=?$

$f(?)=2$

0

$f(0)=\cancel{0^3}^{\,0}+3\cancel{\sin 0}^{\,0}+2\cancel{\cos 0}^{\,1}$

$f(0)=2$

we learned

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}$$

$$=\frac{1}{f'(0)}$$

$$=\frac{1}{3\cancel{(0)^2}^{\,0}+3\cancel{\cos 0}^{\,1}-2\cancel{\sin 0}^{\,0}}=\frac{1}{3}$$

Recap:

$$\boxed{(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}}$$

working with exponential functions:

$f(x) = e^x$

find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x e^h - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{\to 1}$$

what is e?

e is when

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$$= e^x \cdot 1 = e^x$$

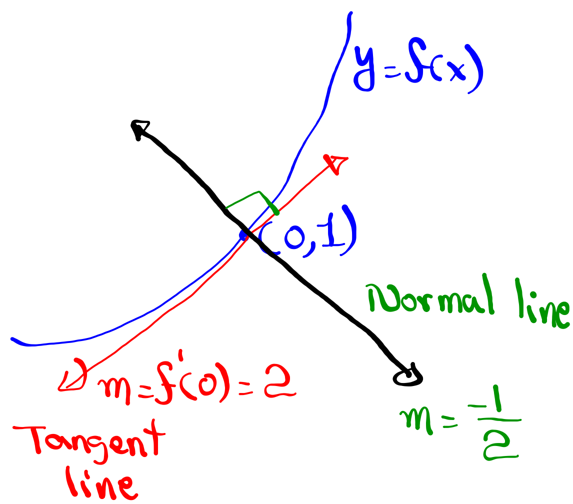
$$\boxed{\frac{d}{dx}\left[e^x\right] = e^x \quad , \quad \int e^x \, dx = e^x + C}$$

$f(x) = x + e^x$

$f(0) = 0 + e^0 = 1$

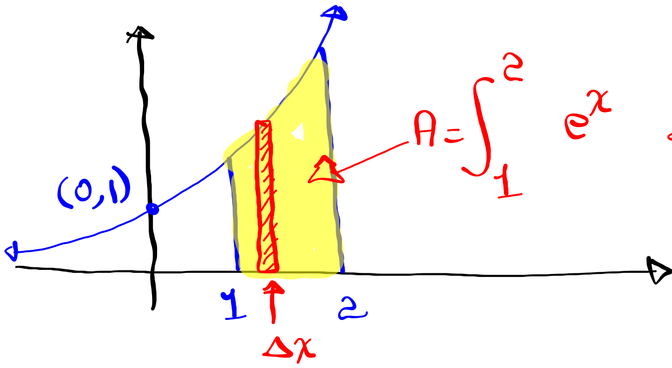
$f'(x) = 1 + e^x$

$f'(0) = 1 + e^0 = 2$



Draw $f(x) = e^x$

shade the region below $f(x)$, above x-axis, between $x=1$ and $x=2$.



$$A = \int_1^2 e^x \, dx = e^x \Big|_1^2$$

$$= e^2 - e^1$$

$$= \boxed{e^2 - e}$$

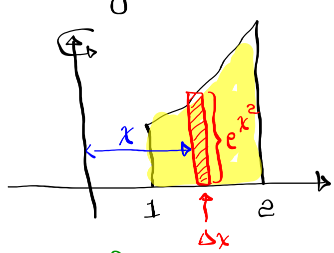
$f(x) = e^{\boxed{x^2}}$

Find $f'(x)$

$f'(x) = e^{x^2} \cdot 2x \qquad f'(x) = 2x e^{x^2}$

Rotate the region enclosed by $f(x) = e^{x^2}$, x-axis, $x=1$, and $x=2$ by Y-axis.

Find the Volume.



Shell

$$V = \int_1^2 2\pi \cdot x \cdot e^{x^2} \, dx$$

$$= \pi \int_1^2 2x \, e^{x^2} \, dx$$

$u = x^2$

$du = 2x\,dx$

$x=1 \to u=1$

$x=2 \to u=4$

$$= \pi \int_1^4 e^u \, du = \pi \, e^u \Big|_1^4 = \boxed{\pi \left[e^4 - e \right]}$$

Evaluate

$$\int e^x \sec^2 e^x \, dx$$

$u = e^x$

$du = e^x dx$

$$= \int \sec^2 u \, du = \tan u + C$$

$$= \boxed{\tan e^x + C}$$

Evaluate

$$\int_1^2 \frac{e^{1/x}}{x^2} dx$$

Hint: $u = \frac{1}{x}$

$du = \frac{-1}{x^2} dx$

$-du = \frac{1}{x^2} dx$

$x=1 \rightarrow u=1$

$x=2 \rightarrow u=\frac{1}{2}$

$$= \int_1^{\frac{1}{2}} e^u \cdot -du$$

$$= -\int_{1/2}^{1} e^u \cdot -du = e^u \Big|_{1/2}^{1} = e^1 - e^{1/2} = \boxed{e - \sqrt{e}}$$